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Time-dependent normal form Hamiltonian for dynamical equilibria

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Abstract. A new time-dependent normal-form procedure for dynamical equilibria (undergoing parametric excitation) of one-dimensional (1D) Hamiltonian systems is developed with the method of Lie transforms. The expansion is based on the Lewis invariant for the linearized motion. The time-dependent Hamiltonian normal form reduces smoothly to the usual representation in the autonomous limit. Illustrative examples of the formalism are focused on time-periodic systems and the dynamics of Hamiltonian switching.

1. Introduction

Autonomous and time-periodic equilibria can be analysed by very similar mathematical tools. This fact is due to Floquet theory for the linearized time-periodic systems, which provides a transformation to an autonomous set of equations [1]. As a consequence, time-periodic Hamiltonian equilibria can be reduced to autonomous normal forms for each stability type.

In this paper we outline an alternative time-dependent normal-form theory, which applies to autonomous, time-periodic, and more general time-dependent equilibria. The development of the canonical formalism relies on the Lewis invariant [2], rather than the less general Floquet theorem (which applies only to time-periodic equilibria). The basic ideas for the first-order normal form of this kind were presented in a study of time-periodic 1D Hamiltonian equilibria and their semiclassical quantization [3, 4]. We also focus on linearly stable equilibria in this paper, however primarily their higher-order normal form. To exemplify the new type of freedom introduced by the formalism we discuss briefly an application to (time-dependent) Hamiltonian switching (see e.g. adiabatic switching [5], microwave ionization of hydrogen [6]).

The explicit form of the time-dependent 1D Hamiltonian is given by

$$H(p,q,t) = h_0(t) + \tfrac{1}{2}p^2 + \tfrac{1}{2}h_2(t)q^2 + \tfrac{1}{3}h_3(t)q^3 + \tfrac{1}{4}h_4(t)q^4 + \cdots \qquad (1)$$

where $h_0(t)$ and $h_n(t), n = 2, 3, \ldots$ are quite arbitrary functions of time. This Hamiltonian with periodic coefficients is typically obtained in the narrow-tube analysis (see [3, 4]), where the coefficients appear periodic due to the underlying periodic centre motion.

We argue in this paper that the Hamiltonian (1) can be reduced to a truncated time-dependent normal form

$$\mathcal{K}_{\mathrm{TD}}(\mathcal{L}_n, t) = \eta_0(t) + \eta_1(t)\mathcal{L}_n + \eta_2(t)\mathcal{L}_n^2 + \eta_3(t)\mathcal{L}_n^3 + \cdots + \eta_n(t)\mathcal{L}_n^n \qquad (2)$$

by applying Lie transforms. In this form of the Hamiltonian, $\mathcal{L}_n$ is an 'approximate' Lewis invariant corresponding to a suitable order n of the canonical expansion. In general, as

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$n \to \infty$, the expansion is expected to be divergent (see the discussion by Wood and Ali [7]). The asymptotic nature is typical also for normal forms of static equilibria. Analysis of the dynamics of the normal form Hamiltonian yields (at least) a qualitative picture of the flows. We recall that the idea of successive canonical transformations to simplify the analysis of a complicated system is the basis of the KAM theory of quasiperiodic motion.

Given the truncated Hamiltonian (2) with periodic coefficients $\eta_n(t+T) = \eta_n(t)$, we may think of the dynamics as motion on a time-periodic tube with constant action $\mathcal{L}_n$ and instantaneous angular velocity

$$\dot{\phi}(\mathcal{L}_n, t) = \eta_1(t) + 2\eta_2(t)\mathcal{L}_n + 3\eta_3(t)\mathcal{L}_n^2 + \cdots + n\eta_n(t)\mathcal{L}_n^{n-1}. \tag{3}$$

From this expression we can directly obtain the characteristic rotation angles in the period map. In the limit $\mathcal{L}_n \to 0$, the winding becomes independent of the action and relates in an exact way to the Floquet exponents of the linearized motion.

For each order of approximation the approximate Hamiltonian also predicts a set of 'level lines', defined by

$$\mathcal{L}_n(p, q, t = \text{constant}) = \mathcal{L}_n \tag{4}$$

which, for a given value of $\mathcal{L}_n$, gives one or more time-dependent closed lines in the two-dimensional phase space. Time-periodic systems can be studied by the period map in which (4) reduces to a polynomial equation of degree $2n$ in p and q, with constant coefficients.

There are some new implications from this normal form expansion. We might, for example, consider quite arbitrary time variations in the Hamiltonian. Specific switching mechanisms can be of interest, as in microwave excitation of Rydberg states [6]. Section 2 deals with the Lewis-invariant representation which is fundamental for the success of the subsequent transformations of the Hamiltonian. In section 3 we outline the arbitrary-order, canonical Lie transformation and discuss some explicit low-order normal-form expressions. By simple examples we illustrate realistic behaviours of key quantities of the theory. We discuss invariance properties of the normal form transformation as well as the study of quasi-energies in periodic systems in section 4. Conclusions are given in section 5.

2. Lewis-invariant representation

The normalization procedure presented here generalizes significantly the one typically used in deriving Birkhoff's normal form (Arnold [1], Ozorio de Almeida [8]). For the leading (harmonic) order the present procedure is outlined in [4]. This corresponds to the present section. For the cubic and quartic orders we proceed to give some explicit formulae in section 3.

To put the quadratic part of the Hamiltonian in a suitable form, we introduce the Lewis invariant (see [9], and more recent comments in Lichtenberg and Lieberman [2]):

$$\mathcal{L}_1 = \tfrac{1}{2}[(p\rho(t) - q\dot{\rho}(t))^2/\Lambda_\mathrm{d} + \Lambda_\mathrm{d}(q/\rho(t))^2]. \tag{5}$$

The Lewis invariant is a non-trivial combination of the canonical variables p, q, and $\rho(t)$, where $\rho(t)$ is a particular solution of the auxiliary Milne equation [10]

$$\ddot{\rho} + h_2(t)\rho = \frac{\Lambda_\mathrm{d}^2}{\rho^3} \tag{6}$$

closely related to the quadratic part of the basic Hamiltonian (1). Equation (6) can be seen as the radial part of the plane-polar components of Newton's second law for a planar mass–spring system having a time-dependent coefficient. The non-uniqueness of the Lewis invariant (discussed in [4]) requires that a particular Milne solution be selected. A general

Milne solution can be expressed in terms of fundamental solutions to the linear equation $\ddot{q} + h_2(t)\, q = 0$ (see sections 1.5 and 1.6 in [11], with the identifications $q(t) = \rho^{-2}(t)$ and $\Lambda = 1$). We consider here solutions that are initially normalized and stationary, satisfying the initial conditions (see Thylwe and Bensch [4]):

$$\rho(0) = 1 \qquad \dot{\rho}(0) = 0. \tag{7}$$

The particular solution is then found by varying the angular momentum parameter Λ (in analogy with polar decompositions for planar central motion problems) until, for a particular value $\Lambda = \Lambda_{\mathrm{d}}$, the Milne solution satisfies some further 'dynamical criterion', a stationarity condition in this presentation. A typical situation is to preserve the (exact) time-periodic symmetry of Hamiltonians that are time periodic. Then, by a proper choice of the initial time, $\Lambda = \Lambda_{\mathrm{d}}$ selects a unique periodic solution $\rho_{\mathrm{p}}(t)$ [4] which returns to its initial turning point (7). For time-switching Hamiltonians, $\Lambda = \Lambda_{\mathrm{d}}$ selects a unique solution that switches between two different 'almost constant limits'. The existence of a 'dynamical Milne solution' for real-valued parameters Λ_{d} is intimately related to the linear stability of the equilibrium and we shall assume that this is the case. A guiding selection principle for more general cases than those considered in the present paper is to modify $\dot{\rho}(0)$ for the normalized Milne solution to keep it as smooth as possible. This will simplify the time dependence of higher-order terms as well.

We adopt the Lewis invariant transformation based on such a 'dynamical Milne solution':

$$(q, p) \to (Q, P) = \left(\frac{\sqrt{\Lambda_{\mathrm{d}}} q}{\rho(t)}, \frac{p\rho(t) - q\dot{\rho}(t)}{\sqrt{\Lambda_{\mathrm{d}}}} \right) \tag{8}$$

and the corresponding time-dependent generating function

$$S(P, q, t) = \frac{\sqrt{\Lambda_{\mathrm{d}}}}{\rho(t)} Pq + \frac{\dot{\rho}(t) q^2}{2\rho(t)}. \tag{9}$$

The transformed Hamiltonian $K(P, Q, t)$ then becomes

$$\begin{aligned} K(P, Q, t) = {} & \eta_0(t) + \tfrac{1}{2}\eta_1(t) \left(P^2 + Q^2\right) \\ & + \tfrac{1}{3} h_3(t) \rho^3(t) \Lambda_{\mathrm{d}}^{-3/2} Q^3 + \tfrac{1}{4} h_4(t) \rho^4(t) \Lambda_{\mathrm{d}}^{-2} Q^4 + \cdots \end{aligned} \tag{10}$$

where

$$\eta_0(t) = h_0(t) \qquad \eta_1(t) = \frac{\Lambda_{\mathrm{d}}}{\rho^2(t)}. \tag{11}$$

In equation (10) new coefficient functions replace the original ones $h_n(t)$, $n = 1, 2, \ldots$. We also recognize the Lewis invariant, $\mathcal{L}_1 = (P^2 + Q^2)/2$, for the quadratic part of the Hamiltonian (the reader may confirm that $\dot{\mathcal{L}}_1 = 0$ for the quadratic part of the Hamiltonian $K(P, Q, t)$) and that $h_2(t)$ is no longer explicitly present. Unfortunately, the higher-order terms are not constants of motion, expressible in terms of the Lewis invariant alone. To handle these we subsequently adopt the complexification (see Arnold [1]) of the Hamiltonian in section 3.

In passing, we notice a particular subclass of nonlinear dynamical equilibria, which are subject to exact integrability (see Kaushal and Korsch [12] and Lewis and Leach [13]). Suppose that the higher-order coefficients $h_n(t)$ in equation (10) are given by

$$h_n(t) = c_n \left(\frac{\Lambda_{\mathrm{d}}}{\rho^2(t)} \right)^{n/2+1} \qquad n \geqslant 3 \tag{12}$$

with constants c_n and the same Milne solution as before. Then all terms of order 2 and higher in the Hamiltonian appear synchronized:

$$K(P, Q, t) = \eta_0(t) + \frac{1}{2}\eta_1(t)\left(P^2 + Q^2 + \frac{2c_3}{3}Q^3 + \frac{2c_4}{4}Q^4 + \cdots\right). \quad (13)$$

Here the expression in the large brackets is an *exact* constant of motion, while the Lewis invariant $\mathcal{L}_1 = (P^2 + Q^2)/2$ is so only for the linearized motion. This peculiar model will not be reported on any further in the present paper.

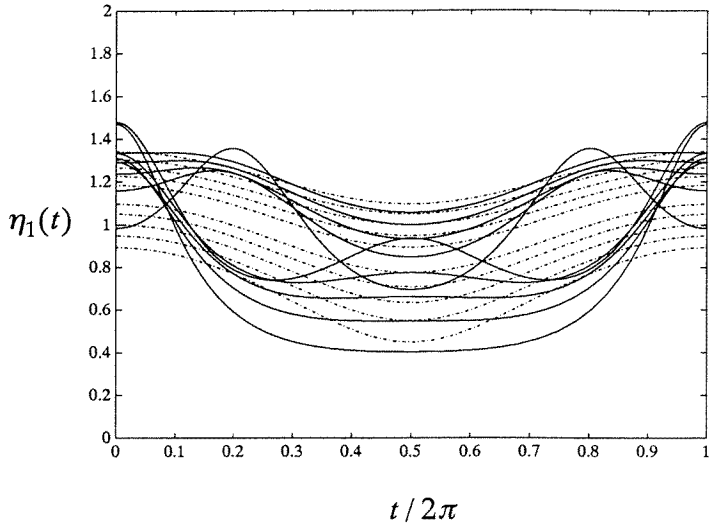

Figure 1. Illustration of the true (——) and adiabatic (— · —) time-periodic behaviour of the leading normal-form coefficient $\eta_1(t)$ when a system parameter A is varied (see example 1). The true curve oscillates about the adiabatic one, and the number of these oscillations changes by one at resonance.

2.1. Adiabatic approximation

The adiabatic limit consists of an approximate Milne solution $\rho(t)$, obtained by assuming that it varies slowly. Hence, simply neglecting the time derivative in Milne's equation we find the leading-order expression

$$\rho_a(t) = [\Lambda_d^2 / h_2(t)]^{1/4}. \quad (14)$$

This means, in particular, that the first non-trivial normal-form coefficient $\eta_1(t)$ becomes

$$\eta_1(t) \approx \sqrt{h_2(t)}. \quad (15)$$

In the following numerical examples we frequently observe that the adiabatic approximation shows smoother behaviour for the coefficient $\eta_1(t)$ than the exact one. Note that the primitive adiabatic approximation fails in cases where the coefficient $h_2(t)$ changes sign.

Example 1. Periodic coefficient $\eta_1(t)$. As an illustration we consider the model $h_2(t) = A + r\cos t$ for the quadratic part of the Hamiltonian (1), for a series of parameter values $A = 0.5, 0.6, \ldots, 0.9, 1.1, 1.2, \ldots, 1.5$ and $r = 0.3$. In figure 1 we plot the numerically calculated coefficient $\eta_1(t) = \Lambda_d/\rho^2(t)$ (full curve) of the transformed quadratic term in (10) and compare it with its adiabatic expression $\sqrt{h_2(t)}$ (chain curve). The exact coefficient

performs oscillations about the adiabatic approximation. At resonance, near $A = 1$ where no periodic Milne solution was found, the number of superimposed non-adiabatic oscillations is changed by one. The amplitudes of these oscillations are smaller for larger values of A.

Example 2. Switching coefficient $\eta_1(t)$. Here we consider the finite-time switching model $h_2(t) = A + r(t - \sin(t))/(2\pi)$, $0 \leqslant t \leqslant 2\pi$. In figure 2 we plot $\eta_1(t) = \Lambda_d/\rho^2(t)$ (full curve) and its adiabatic approximation (chain curve) for a sequence of parameters given by $A = 1$, $r = 0.1, 0.3, \ldots, 1.1, 1.5, 1.7, \ldots, 2.1$. In contrast to the adiabatic ones, the exact coefficients $\eta_1(t)$ generally show a non-monotonic behaviour during the switching process due to non-adiabatic effects. A strong non-adiabatic resonance effect in the parameter range $1.2 \leqslant r \leqslant 1.4$ is clearly visible.

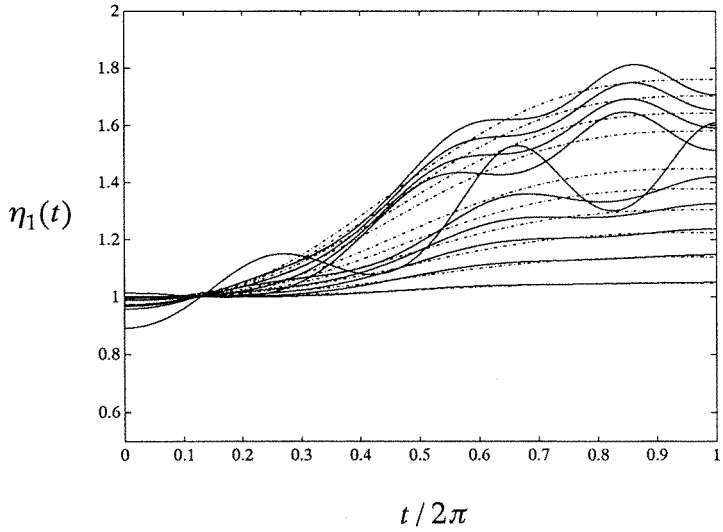

Figure 2. Illustration of the true (——) and adiabatic ($-\cdot-\cdot$) switching behaviour of the leading normal-form coefficient $\eta_1(t)$ when a system parameter r is varied (see example 2). Again the true curve oscillates about the adiabatic one and resonances may occur.

3. Complexification and Lie transforms

To simplify the higher-order terms we now complexify the Hamiltonian (10) by introducing the momentum $z = P + \mathrm{i}Q$ and its conjugate variable $z^* = P - \mathrm{i}Q$ (cf [1]). The resulting Hamiltonian is

$$K^{\mathrm{c}}(z, z^*, t) = -2\mathrm{i}K(P(z, z^*), Q(z, z^*), t) \tag{16}$$

so that

$$\begin{aligned} K^{\mathrm{c}}(z, z^*, t) = {} & -2\mathrm{i}\eta_0(t) - \mathrm{i}\eta_1(t) z z^* - 2\mathrm{i}b_3(t)\left(-\tfrac{1}{2}\mathrm{i}\right)^3 (z - z^*)^3 \\ & -2\mathrm{i}b_4(t)\left(-\tfrac{1}{2}\mathrm{i}\right)^4 \left(z - z^*\right)^4 - \cdots . \end{aligned} \tag{17}$$

Here we have simplified the notation for the coefficient functions in the Hamiltonian (10) by introducing

$$b_n(t) = \frac{1}{n} h_n(t) \rho^n(t) \Lambda_{\mathrm{d}}^{-n/2} \qquad n = 3, 4, \ldots . \tag{18}$$

The reader may confirm that z and z^* satisfy the canonical equations for the Hamiltonian $K^{\mathrm{c}}(z, z^*, t)$, if P and Q are canonical variables for the Hamiltonian $K(P, Q, t)$. Furthermore, for real physical variables, z and z^* stay complex conjugates and $K^{\mathrm{c}}(z, z^*, t)$ is always imaginary.

For later ease of notation, we write K^{c} as

$$K^{\mathrm{c}} = K_0^{\mathrm{c}} + K_1^{\mathrm{c}} + K_2^{\mathrm{c}} + K_3^{\mathrm{c}} + \cdots \tag{19}$$

where K_0^{c} is the quadratic part of the Hamiltonian and K_n^{c} are monomials of degree $n+2$. In particular, the subsequent manipulations will leave K_0^{c} unchanged.

We will now seek a canonical transformation, so as to eliminate as many terms as possible in the new Hamiltonian $\mathcal{K}^{\mathrm{c}}$. For this purpose we turn to the method of Lie transforms. In particular, we seek a function $w(z, z^*)$, known as the *Lie generating function*, which is used to generate a Hamiltonian system via the canonical equations

$$\frac{\mathrm{d}Z}{\mathrm{d}s} = \{Z, w\} = -\frac{\partial w}{\partial Z^*} \qquad \text{and} \qquad \frac{\mathrm{d}Z^*}{\mathrm{d}s} = \{Z^*, w\} = \frac{\partial w}{\partial Z} \tag{20}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket with respect to (Z^*, Z), and s is a formal, real 'evolution' parameter for the transformation. The solution to this system with $Z(s=0) = z$ and $Z^*(s=0) = z^*$ constitutes a canonical transformation for every value of s. Because of the complex symmetry of the canonical variables in (20), we have

$$w^* = -w \tag{21}$$

which in turn means that w is imaginary. For further details we refer the reader to Lichtenberg and Lieberman [2] and a more tutorial presentation by Cary [14]. To proceed, we express w as the formal expansion series ($s = 1$)

$$w = \sum_{n=1}^{\infty} w_n \tag{22}$$

where w_n are sums of monomials of degree $n+2$, i.e

$$w_n = \sum_{k+l=n+2} a_{kl} z^k z^{*l} \tag{23}$$

with time-dependent coefficients a_{kl} to be determined in the transformation process. For notational simplicity we define the operators

$$L_n = \{w_n, \} \tag{24}$$

$$T_n = -\frac{1}{n} \sum_{m=0}^{n-1} T_m L_{n-m} \tag{25}$$

$$T_n^{-1} = \frac{1}{n} \sum_{m=0}^{n-1} L_{n-m} T_m^{-1} \tag{26}$$

where $T_0 = 1$ and $T_0^{-1} = 1$. The canonical transformation obtainable through formal expansion of equation (20) is now given by

$$z = \sum_{n=0}^{\infty} T_n^{-1} Z \qquad \text{and} \qquad z^* = \sum_{n=0}^{\infty} T_n^{-1} Z^* \tag{27}$$

and, similarly,

$$Z = \sum_{n=0}^{\infty} T_n z \qquad \text{and} \qquad Z^* = \sum_{n=0}^{\infty} T_n z^*. \tag{28}$$

Following [14], the classical (mixed variable) relation $K = H + \partial F/\partial t$, where F is a generating function, is now replaced (in expanded form) by the fundamental transformation equation

$$\frac{\partial}{\partial t} w_n + \{w_n, K_0^{\mathrm{c}}\} = n\mathcal{K}_n^{\mathrm{c}} - \left[nK_n^{\mathrm{c}} + \sum_{m=1}^{n-1} (L_{n-m}\mathcal{K}_m^{\mathrm{c}} + mT_{n-m}^{-1} K_m^{\mathrm{c}}) \right] \tag{29}$$

which can be used to find the necessary transformation from the original Hamiltonian K^{c} and the desired form of the transformed Hamiltonian $\mathcal{K}^{\mathrm{c}}$. It is assumed here that $\mathcal{K}_0^{\mathrm{c}} = K_0^{\mathrm{c}}$. For each n we seek w_n which satisfy equation (29) where $\mathcal{K}_n^{\mathrm{c}}$ contains as few non-vanishing terms as possible.

Let us for the moment assume that $\mathcal{K}_n^{\mathrm{c}} = 0, n > 0$. Since equation (29) is a sum of monomials of degree $n+2$ in the dummy variables z and z^* (cf Lichtenberg and Lieberman [2]) the known time-dependent coefficients on the right-hand side may collectively be written as $2\mathrm{i} f_{kl}(t)$, $k+l = n+2$. We then find the following differential equations for the coefficients a_{kl}:

$$\dot{a}_{kl} - \mathrm{i}\eta_1(t)(l-k)a_{kl} = 2\mathrm{i} f_{kl}(t). \tag{30}$$

These can formally be solved to yield

$$a_{kl}(t) = \left[a_{kl}(0) + 2\mathrm{i} \int_0^t f_{kl}(s) \exp\left[-\mathrm{i}(l-k) \int_0^s \eta_1(x)\,\mathrm{d}x \right] \mathrm{d}s \right] \exp\left[\mathrm{i}(l-k) \int_0^t \eta_1(s)\,\mathrm{d}s \right]. \tag{31}$$

For a large class of bounded functions $f_{kl}(t)$ and $\eta_1(t)$ we expect (but we have no proof of this statement) bounded solutions of equation (30) as long as $l \neq k$. In particular, for periodic coefficients, the existence of periodic solutions is given by the sufficient non-resonance condition

$$\int_0^T \eta_1(t)\,\mathrm{d}t \neq \frac{2\pi j}{l-k} \tag{32}$$

for all integers j, k and $l \neq k$, such that $k + l = n + 2$. This is realized by equating the expression for $a_{kl}(T)$, using (31), with the initial value $a_{kl}(0)$ and then algebraically solving for $a_{kl}(0)$.

In the general case when $l = k$ there is no guarantee that a bound solution can be found (unless $f_{kk}(t) = 0$). It is then convenient, but by no means necessary, to set $a_{kk} = 0$ in the Lie generating function, in which case the left-hand side of (30) vanishes. Instead we have to admit a non-vanishing diagonal term in the Hamiltonian $\mathcal{K}_n^{\mathrm{c}}$ that cancels the existing diagonal terms ($2\mathrm{i} f_{kk}(t)$) in the right-hand member of the fundamental equation (29). In fact, in the case of periodic coefficients, it is still possible to eliminate all oscillating parts of f_{kk} by a coefficient function $a_{kk}(t)$ that does not vanish.

In summary, all non-diagonal terms in the expression for K_n^{c} can be eliminated in the transformed Hamiltonian. For the diagonal cases, we obtain the expression

$$\mathcal{K}_n^{\mathrm{c}} = -2\mathrm{i}(f_{kk}(t)/n) Z^k Z^{*k} \qquad n = 2k - 2. \tag{33}$$

As previously noted, this expression can be modified somewhat in specific situations.

From the above, we immediately conclude that all monomials of odd degree can be eliminated from the Hamiltonian, since they contain no diagonal terms. Thus, with a new index $m = (n-2)/2$ labelling the remaining terms, the general truncated expression for the transformed Hamiltonian is

$$\mathcal{K}^{\mathrm{c}}(Z, Z^*, t) = -2\mathrm{i} \sum_{m=0}^{M} \eta_m(t) \left(\frac{ZZ^*}{2} \right)^m. \tag{34}$$

The coefficients $\eta_m(t)$ contain the corresponding diagonal terms of the original Hamiltonian and 'previously determined' coefficients of the generating function.

3.1. Cubic terms

For $n = 1$, the right-hand side of equation (29) contains only the cubic terms of the original Hamiltonian (17) with opposite sign. Hence, we obtain the corresponding monomial coefficients

$$f_{3-l,l}(t) = b_3(t)\left(-\frac{\mathrm{i}}{2}\right)^3\binom{3}{l}(-1)^l. \tag{35}$$

This expression can be substituted into equation (31) to yield explicit solutions for the transformation coefficients. This implies that all terms in $\mathcal{K}_1^{\mathrm{c}}$ can consistently be set to zero. Note also the specific symmetry $f_{3-l,l}(t) = f^*_{l,3-l}(t)$, which from equation (30) is consistent with the symmetry

$$a_{3-l,l} = -a^*_{l,3-l} \tag{36}$$

for the transformation coefficients (see also equation (21)).

3.2. Quartic terms

Similarly, for $n = 2$, equation (29) becomes

$$\frac{\partial}{\partial t}w_2 + \{w_2, K_0^{\mathrm{c}}\} = 2\mathcal{K}_2^{\mathrm{c}} - [2K_2^{\mathrm{c}} + \{w_1, K_1^{\mathrm{c}}\}] \tag{37}$$

where the left-hand sides of this equation and that of (30) look identical for each monomial (here $k = 4 - l$).

Using the terms on the right-hand side of (37) which are known at this stage, i.e.

$$K_1^{\mathrm{c}} = -2\mathrm{i}b_3(t)\left(-\tfrac{1}{2}\mathrm{i}\right)^3(z - z^*)^3 \tag{38}$$

$$K_2^{\mathrm{c}} = -2\mathrm{i}b_4(t)\left(-\tfrac{1}{2}\mathrm{i}\right)^4(z - z^*)^4 \tag{39}$$

and

$$w_1 = \sum_{l=0}^{3} a_{3-l,l}z^{3-l}z^{*l} \tag{40}$$

we identify monomials from the equation

$$2\mathrm{i}\sum_{l=0}^{4} f_{4-l,l}z^{4-l}z^{*l} = \tfrac{1}{4}\mathrm{i}b_4(z - z^*)^4 - \tfrac{3}{4}b_3(z - z^*)^2((a_{21} + 3a_{30})z^2 + 2\,(a_{12} + a_{21})zz^* + (3\,a_{03} + a_{12})z^{*2}). \tag{41}$$

The symmetry for the cubic coefficients again supports the symmetries $f_{4-l,l} = f^*_{l,4-l}$ and $a_{4-l,l} = -a^*_{l,4-l}$ in (30). Hence, for off-diagonal terms we need to solve only two equations, e.g.,

$$\dot{a}_{40} + 4\mathrm{i}\eta_1(t)\,a_{40} = \tfrac{1}{4}\mathrm{i}b_4 - \tfrac{1}{4}b_3(9a_{30} + 3a_{12}) \tag{42}$$

and

$$\dot{a}_{31} + 2\mathrm{i}\eta_1(t)\,a_{31} = -\mathrm{i}b_4 + \tfrac{1}{2}b_3(9a_{30} - 3a_{12}). \tag{43}$$

For the diagonal terms we put $a_{22} = 0$ and cancel the right-hand side of (37) by the aid of (33). This leaves us with a contribution to the transformed Hamiltonian

$$\mathcal{K}_2^c = -\mathrm{i}\tfrac{3}{4}\left(b_4(t) + \tfrac{3}{2}\mathrm{i}b_3(t)[(a_{30} + a_{03}) - (a_{21} + a_{12})]\right)(ZZ^*)^2. \tag{44}$$

With reference to the final expression for the transformed Hamiltonian, we are now able to read off the relevant coefficient $\eta_2(t)$. A final use of the complex symmetry (36) yields

$$\eta_2(t) = \tfrac{3}{2}b_4(t) - \tfrac{9}{2}b_3(t)(\mathrm{Im}\ a_{03} - \mathrm{Im}\ a_{12}). \tag{45}$$

This expression for the coefficient $\eta_2(t)$ indicates a significant contribution from the cubic terms of the original Hamiltonian that has not previously been taken into account in the narrow-tube quantization of time-periodic systems [3].

We again remark that, in the case of time-periodic coefficients, the oscillating part of the right-hand side of equation (44) can be eliminated, and η_2 then equals the average of the right-hand side. This procedure can be carried over to all even polynomial orders with the result that η_m =constant for all $m > 1$ in equation (34).

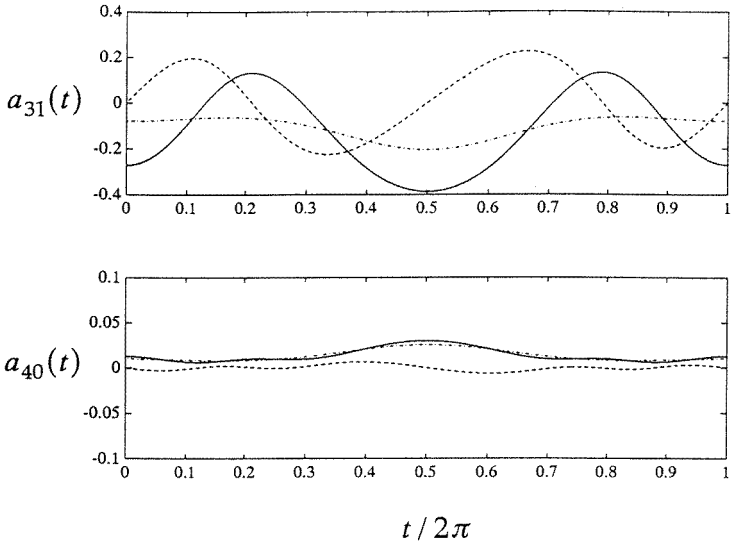

Figure 3. Illustration of the time-periodic behaviour of the large and small (complex) coefficient functions $a_{31}(t)$ and $a_{40}(t)$, respectively, for the model considered in example 3. The true (——) and adiabatic (— · —) behaviour are compared. Note that the adiabatic approximation is real valued, whilst the true value of the coefficient has an imaginary component (- - - -).

3.3. Adiabatic approximations

An interesting point is to consider the adiabatic approximations of equation (30). They are indeed of the same type as the coefficients $\eta_0(t)$ and $b_3(t)$ and easy to calculate.

The time-dependent coefficients a_{kl} of w_1 and w_2 allow the adiabatic approximations

$$a_{3-k,k}(t) = -\mathrm{i}\frac{(-1)^k}{4(2k-3)}\binom{3}{k}\frac{b_3(t)}{\eta_1(t)} \tag{46}$$

and

$$a_{4-l,l}(t) = \frac{(-1)^l}{2(4-2l)}\binom{4}{l}\frac{b_4(t)}{\eta_1(t)} + F(w_1, H_1, \eta_1) \qquad \text{for } l \neq 2 \tag{47}$$

provided ($\dot{a}(t)/a(t) \ll 1$). The non-homogeneity term F is linear in the coefficients of w_1.

The validity of the adiabatic approximation for the a-functions is briefly considered in the numerical examples.

Example 3. Periodic case. We extend the quadratic periodic model (see example 1) $h_2(t) = A + r\cos t$, with $A = 1.2, r = 0.3$, to include $h_4(t) = 1$ (constant) in the Hamiltonian (1). Hence, this model does not have any cubic terms. Four periodic off-diagonal a-coefficients can be determined ($l = 0, 1, 3, 4$), but the last two are related to the first two by the symmetry $a_{4-l,l} = -a^*_{l,4-l}$. The upper part of figure 3 shows the real (full curve) and imaginary (broken curve) parts of $a_{31}(t)$ together with the (real) adiabatic approximation (chain curve). The lower part shows the same quantities for the smaller $a_{40}(t)$. The adiabatic approximation seems to be poor for the larger coefficient function $a_{31}(t)$.

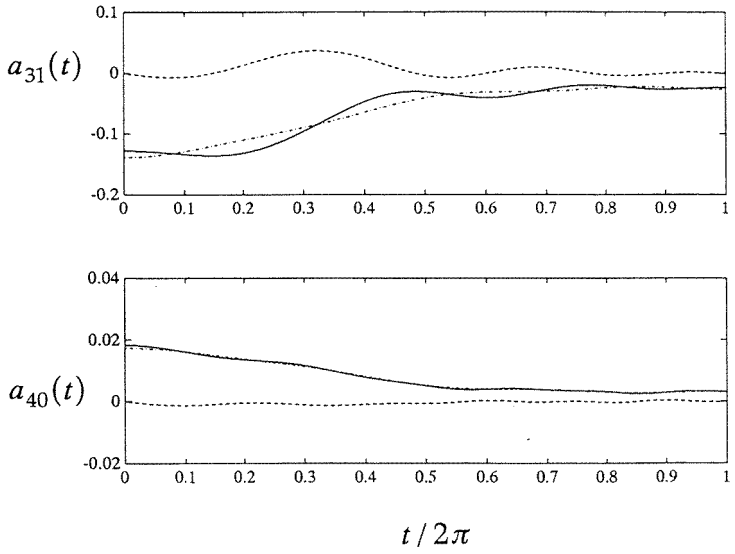

Figure 4. Illustration of the (complex) coefficient functions $a_{31}(t)$ and $a_{40}(t)$ and their (real-valued) adiabatic approximations for the switching model considered in example 4. The curves are identified as in figure 3.

Example 4. Switching case. We extend the quadratic model in example 2 with $A = 1, r = 2$ to include $h_4(t) = 1$ (constant). Two off-diagonal (and non-zero) a-coefficients are to be determined ($l = 0, 1$). The criterion (for this example) is that no imaginary parts exist at the endpoints (since the adiabatic solution is real): this corresponds to stationary absolute value of the complex a's, which can be derived from the differential equations (30) for the a's. One can show that

$$[\dot{|a_{kl}|^2}] = 2\,\mathrm{Re}\,(2\mathrm{i}f_{kl}(t)a^*_{kl}) \tag{48}$$

and the inhomogeneity, shortly denoted $2\mathrm{i}f_{kl}(t)$ as in (30), is always imaginary in our example. The solutions which are real and stationary (in the above sense) at the endpoints are found with $\rho(0) = 1.017\,86$, $a_{31}(0) = -0.128\,17$, $a_{31}(T) = -0.023\,44$, $a_{40}(0) = 0.018\,332$, and $a_{31}(T) = 0.003\,17$. The top part of figure 4 shows the real (full curve) and imaginary (broken curve) parts of $a_{31}(t)$ together with the (real) adiabatic approximation (chain curve). The bottom part of the figure shows the same quantities for $a_{40}(t)$.

4. Time-dependent normal form

4.1. Invariant properties

While the primary purpose of the normal form transformation is to obtain an integrable, albeit time-dependent, truncated Hamiltonian, it is in some cases of interest to preserve certain symmetries or characteristic behaviour of the coefficients of the Hamiltonian. For example, we have already argued that in the case of time-periodic coefficients a normal form transformation can be found which preserves this property. Consequently, the techniques appropriate to the study of time-periodic systems are still applicable to the transformed Hamiltonian.

Along the same lines is the possibility of preserving invariance under time reversal. Thus, assuming that $h_n(-t) = h_n(t)$ for all n, it follows that a symmetric solution can be found to the Milne equation by requiring $\dot{\rho}(0) = 0$. Hence, the same is true of $\eta_1(t)$, as well as the b_n's.

Now consider the evolution equations for the transformation coefficients, equation (30). It is easy to show that if the right-hand side is invariant under the composition of time reversal and complex conjugation, then it is possible to find a solution satisfying $a_{kl}(-t) = -a_{kl}^*(t)$. Similarly, if the right-hand side changes sign under the above operations, solutions satisfying $a_{kl}(-t) = a_{kl}^*(t)$ can be found. We assume that these solutions are chosen for the normal form transformation.

Using equation (29) and the definitions of the operators L_i and T_i^{-1} one can show that for $k + l =$ odd, the right-hand side remains invariant, while for $k + l =$ even it changes sign. The proof is one of induction, noting from equation (35) that the right-hand side is real for $k + l = 3$. Hence, for all orders we have $a_{kl}(-t) = (-1)^{k+l} a_{kl}^*(t)$.

To show that the transformed Hamiltonian respects the time reversal symmetry, we recognize that only the diagonal terms in the bracketed expression in equation (29) are retained. It is easy to show that the first term is invariant under time reversal. Further, the first term in the sum can be shown to result in no diagonal terms. We now write the second term in the sum as $g(a_{kl}(t), t)$, where we omit the dependence on z and z^*. g is clearly symmetric in the second argument. We will now use the symmetry of the previous paragraph and that implied by equation (21) ($a_{kl} = -a_{lk}^*$). One can then show that

$$g(a_{kl}(-t), -t) = g((-1)^{k+l} a_{kl}^*(t), t) = g((-1)^{k+l+1} a_{lk}(t), t).$$

Again, using induction, it is easy to show that the substitution implied by the last expression is equivalent to the substitution $z \leftrightarrow z^*$ in the original expression. However, since the diagonal terms are unaffected by this latter substitution, they are similarly invariant under time reversal, thus proving our assertion.

We finally note that for other characteristic time dependences, such as the finite-time switching considered in the examples, it is possible that the properties can be preserved in some general sense. However, for finite-time studies, such invariance properties may not be of central concern, since the integrability of the transformed Hamiltonian is the fundamental aim of the analysis.

4.2. The Lewis-invariant formulation

We now return to the Lewis invariant representation. In terms of a particular new Lewis invariant $2\mathcal{L}_M = ZZ^*$, consistent with the order of approximation, we can express the

Hamiltonian $\mathcal{K}_{\rm TD} = \mathcal{K}^{\rm c}/(-2{\rm i})$ as follows:

$$\mathcal{K}_{\rm TD} = \sum_{m=0}^{M} \eta_m(t)\mathcal{L}_M^m. \tag{49}$$

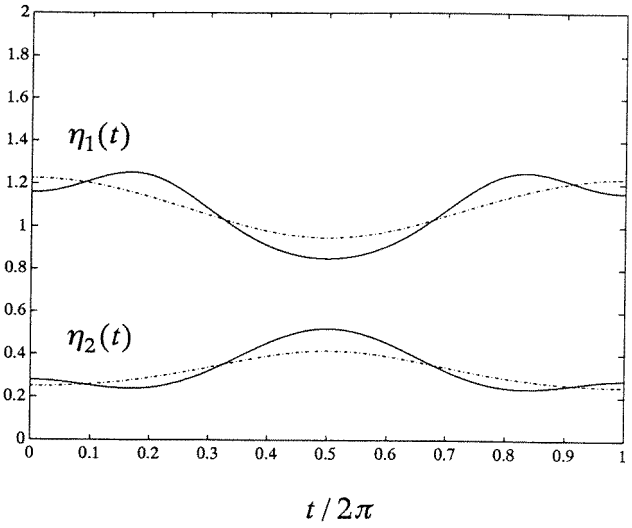

Figure 5. Illustration of the two periodic coefficient functions $\eta_1(t)$ and $\eta_2(t)$ (——) and their adiabatic approximations (— · —) remaining in the normal-form Hamiltonian. The model system is described in example 5.

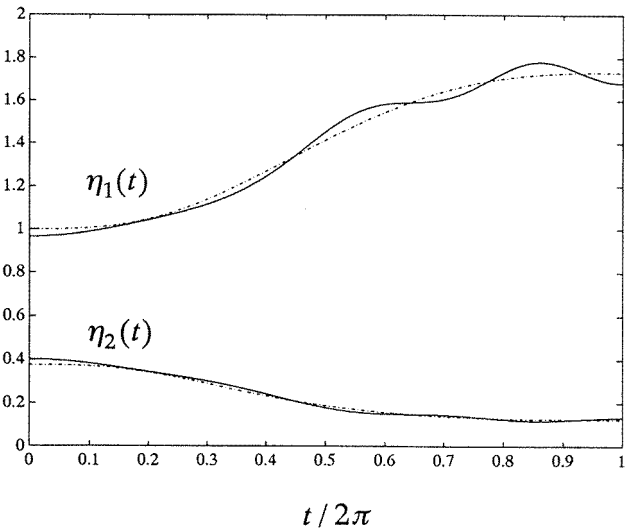

Figure 6. Illustration of the two switching coefficients $\eta_1(t)$ and $\eta_2(t)$ (——) and their adiabatic approximations (— · —) remaining in the normal-form Hamiltonian. The model system is described in example 5.

It is convenient to re-introduce Cartesian variables $\mathcal{P}_M$ and $\mathcal{Q}_M$, corresponding to the invariant $\mathcal{L}_M$, through

$$\mathcal{P}_M = (Z + Z^*)/2 \qquad \mathcal{Q}_M = (Z - Z^*)/(2\mathrm{i}) \tag{50}$$

so that

$$\mathcal{L}_M = (\mathcal{P}_M^2 + \mathcal{Q}_M^2)/2. \tag{51}$$

With these variables the level lines of constant $\mathcal{L}_M$ are always plain circles.

Example 5. Periodic and switching cases. With the same quartic models of the Hamiltonian (1) as earlier in examples 3 and 4 we study the magnitudes of the remaining diagonal normal-form coefficients. In figures 5 and 6, both $\eta_1(t)$ and $\eta_2(t)$ are calculated in two ways: with the exact Milne solution (full curve) and with the adiabatic approximation (chain curve).

4.3. Quasi-energies and level lines

Particularly for time-periodic Hamiltonian systems, and motion on periodic vortex tubes, one can define a characteristic (quasi-) energy, which is a time-averaged value of the proper Hamiltonian. The quasi-energy has a direct relation to quantum mechanics (see [4]). If the Hamiltonian is of the time-periodic normal-form type the integration is straightforward. Hence,

$$\epsilon(\mathcal{L}_M) = \frac{1}{T}\int_0^T \mathcal{K}_{\mathrm{TD}}(\mathcal{L}_M, t)\,\mathrm{d}t' \tag{52}$$

implying that

$$\epsilon(\mathcal{L}_M) = \langle\eta_0\rangle_t + \langle\eta_1\rangle_t \mathcal{L}_M + \langle\eta_2\rangle_t \mathcal{L}_M^2 + \cdots. \tag{53}$$

Quantum mechanically, of course, the constant of motion $\mathcal{L}_M$ is quantized and has to be labelled by a quantum number (note that M is related to the order of transformation—not a quantum number), as it represents the autonomous harmonic oscillator energy in $(\mathcal{R}_M, \mathcal{S}_M)$ phase space.

We mentioned earlier that the level lines corresponding to a sequence of invariants $\mathcal{L}_M$ are plain circles of radii $\sqrt{2\mathcal{L}_M}$ in the transformed phase space. It is also interesting, but tedious, to study the level lines in the original phase space. Not only will the shapes be different, but most importantly, they become time dependent. The canonical transformations just guarantee the same phase space area, $2\pi\mathcal{L}_n$, circumvented by the closed lines. If one is not particularly interested in the time dependence of the level lines, one can study them in a particular period map corresponding to $t = 0, T, \ldots$. This restriction further simplifies the reversion of the coordinate transformations if $\dot{\rho}(0) = 0$.

Example 6. Quasi-energies and level lines. In order to obtain the level lines of the higher-order invariants one has to use the transformation (28), expressing the higher-order invariant $ZZ^*/2$ in terms of the lower-order one $zz^*/2$ and further monomials of lower order. Finally the real canonical coordinates are substituted into the expression. For the particular quartic model studied in the above series of examples we find the following equation for the level lines on the initial-time surface of the section:

$$\mathcal{L}_2 = \left[p^2 r_0^6 + 2\,a r_0^8 p^4 - 2\,q^2 r_0^4 c p^2 + q^2 r_0^2 + 2\,b q^4\right]/(2\,r_0^4) \tag{54}$$

Table 1. Analysis of phase-space level lines of the periodic model. Numerical comparison of first and second normal-form approximations.

$q(0)$	$\mathcal{L}_1$	ϵ_1	$\mathcal{L}_2$	ϵ_2
0.05	1.4486(−3)	1.5786(−3)	1.4510(−3)	1.5819(−3)
0.10	5.7942(−3)	6.3135(−3)	5.8340(−3)	6.3695(−3)
0.15	1.3037(−2)	1.4205(−2)	1.3238(−2)	1.4490(−2)
0.20	2.3177(−2)	2.5254(−2)	2.3814(−2)	2.6160(−2)
0.25	3.6214(−2)	3.9459(−2)	3.7768(−2)	4.1688(−2)
0.30	5.2148(−2)	5.6821(−2)	5.5371(−2)	6.1483(−2)
0.35	7.0979(−2)	7.7340(−2)	7.6950(−2)	8.6066(−2)

where

$$r_0 = \rho(0)/\sqrt{\Lambda_{\rm d}} \tag{55}$$

$$a = 2a^R_{40}(0) + a^R_{31}(0) \tag{56}$$

$$b = 2a^R_{40}(0) - a^R_{31}(0) \tag{57}$$

$$c = 12a^R_{40}(0). \tag{58}$$

To obtain (58) we have made use of index symmetries of the real and imaginary parts of the a coefficients. Recall also that $a_{22} = 0$.

In a numerical study of level lines in the initial ($t = 0, 2\pi, \ldots$) period map, we have chosen a sequence of initial conditions $q(0)$ with $p(0) = 0$ and analysed the first and second Lewis invariant together with the corresponding quasi-energies in the first- and second-order normal-form approximation. The results are collected in table 1. The two sets of approximate level lines are shown in figures 7 and 8.

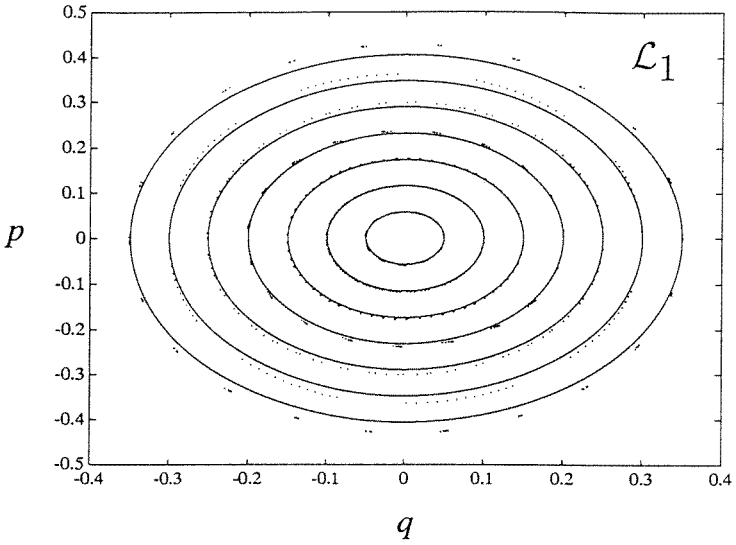

Figure 7. Phase-space level lines corresponding to the first-order Lewis invariant $\mathcal{L}_1$. The numerical period map of the trajectories is shown as small dots. The model is described in example 6.

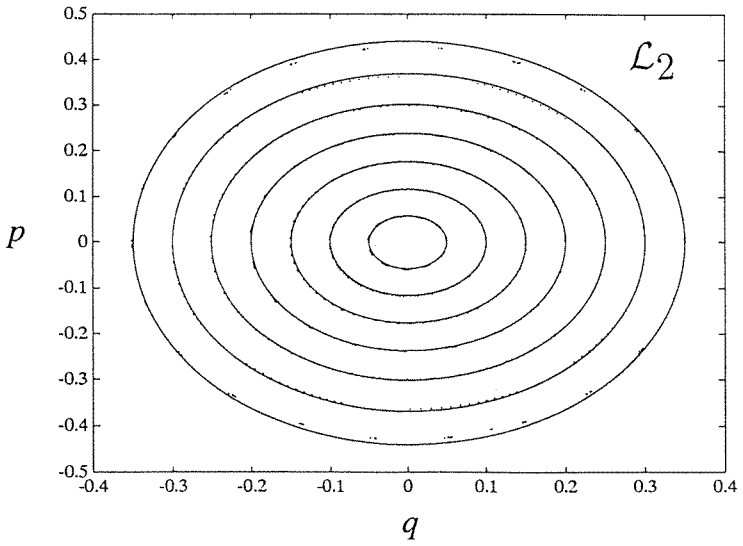

Figure 8. Phase-space level lines for the same model as in figure 7 corresponding to the second-order Lewis invariant $\mathcal{L}_2$.

5. Conclusions

We have developed the basis of a Hamiltonian normal-form approximation for dynamical equilibria that reveals physical insights of system behaviour in between period maps and switching processes. In a numerical study of two model systems we have observed non-adiabatic effects of the dynamics which are typically related to a kind of 'linear' resonance in canonical adiabatic theory [2].

In Hamiltonian switching processes, we also found non-adiabatic effects as well as instabilities of the representation. In this normal form, and in its adiabatic approximation, initial and final energies are effectively connected. In some parameter regions the non-adiabatic effects cause the initial and final energies to shift in the same direction, thereby maintaining an energy difference which is very similar to the one predicted by adiabaticity. However, as we demonstrated in example 2, non-adiabatic effects can in neighbouring parameter regions shift initial and final energies in opposite directions. One may thus introduce 'in-phase' and 'out-of phase' regions of parameter space. These regions are separated by a kind of non-adiabatic resonance for which the formalism breaks down.

In this presentation we have not considered any nonlinear resonance phenomena typical for time-periodic systems. Relevant generalizations to cover uniform descriptions of subharmonic resonances are in progress. Nor have we studied in detail the stability conditions for the Hamiltonian switching case, or the convergence aspects of the expansion in general.

Acknowledgments

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